Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Minors of Axi-symmetric Determinants.

By E. J. Nanson.

The linear relations between certain minors of any axi-symmetric determinant which are due to Kronecker, and certain extensions thereof due to Metzler and Muir,[*] are connected in a remarkably simple way with certain general determinant theorems due to Schweins and Sylvester. It is the object of this paper to establish the connection referred to and also to prove a general theorem which includes the whole of the relations mentioned as particular cases.

1. It has been shown long ago by Clebsch that any axi-symmetric determinant $|a_{pq}|$, where $p, q = 1, 2, \ldots. n$, which, in Sylvester's umbral notation, is denoted by

$$\begin{pmatrix} 1 & 2 & 3 \ldots. n \\ 1 & 2 & 3 \ldots. n \end{pmatrix},$$

may be represented symbolically by

$$\frac{1}{n!} (abc \ldots.)^2,$$

where

$$a_{pq} = a_p a_q = b_p b_q = c_p c_q = \text{etc.},$$

---

[*] Phil. Mag., April, 1902, pp. 410-416.

and $(abc \dots)$ is an abbreviation for the determinant of order n whose r^{th} column is

$$a_r,\ b_r,\ c_r, \text{ etc.},$$

and the symbols a_p, b_p, etc., obey the fundamental laws of algebra, so that

$$a_{pq} = a_p a_q = a_q a_p = a_{qp},$$

as required on account of the symmetry assumed; and, consequently, a general determinant cannot be represented symbolically by Clebsch's method. Nevertheless, any minor of an axi-symmetric determinant can be so represented. Thus the minor formed with the r rows $\alpha, \beta, \gamma, \dots \kappa$ and the r columns $\alpha', \beta', \gamma', \dots \kappa'$ of $|a_{pq}|$ is represented in Sylvester's umbral notations by

$$\begin{pmatrix} \alpha & \beta & \gamma & \dots & \kappa \\ \alpha' & \beta' & \gamma' & \dots & \kappa' \end{pmatrix}$$

and may be represented in Clebsch's symbolic notation by

$$\frac{1}{r!}(a_\alpha b_\beta c_\gamma \dots k_\kappa)(a_{\alpha'} b_{\beta'} c_{\gamma'} \dots k_{\kappa'}),$$

where $(a_\alpha b_\beta c_\gamma \dots k_\kappa)$ is an abbreviation for the determinant of order r formed with the r columns $\alpha, \beta, \gamma, \dots \kappa$ of the array whose p^{th} column is

$$a_p,\ b_p,\ \dots\ k_p,$$

where there are r letters $a, b, \dots k$.

2. As shown by Clebsch, calculations involving the elements a_{pq} may be carried out by means of the symbols a_p, b_p, etc., provided care is taken not to introduce any power of any of these symbols higher than the second. It follows, therefore, that from any homogeneous quadratic relation connecting determinants of the type

$$(a_\alpha b_\beta c_\gamma \dots k_\kappa).$$

where there are r letters $a, b, c, \dots\dots k$, and the letters $\alpha, \beta, \gamma, \dots\dots \varkappa$ may be any r numbers selected from $1, 2, 3, \dots\dots n$, we can at once deduce a linear relation connecting minors of order r of any axi-symmetric determinant.

3. To obtain a quadratic relation of the kind mentioned, let $\alpha, \beta, \gamma, \dots \varkappa$, $\alpha', \beta', \gamma', \dots\dots \varkappa'$ be any two sets, each consisting of r numbers, chosen from $1, 2, 3, \dots\dots n$. Let Δ denote the array whose p^{th} column is

$$a_p, b_p, c_p, \dots\dots k_p,$$

where there are r letters $a, b, c, \dots\dots k$. Also, let λ, μ, etc., be sets chosen from $1, 2, 3, \dots\dots n$, and let the arrays, square or rectangular as the case may be, formed with the sets of columns λ, μ, etc., of Δ, be denoted for brevity by $(\lambda), (\mu)$, etc. Now, specializing, let λ, μ be complementary sets chosen from $\alpha, \beta, \gamma, \dots \varkappa$, and let λ', μ' be complementary sets chosen from $\alpha', \beta', \gamma', \dots \varkappa'$. We have then to consider the determinant

$$\begin{vmatrix} (\lambda) & (\mu) & (\lambda') & (0) \\ (0) & (\mu) & (\lambda') & (\mu') \end{vmatrix} \tag{1}$$

of order $2r$, where (0) denotes an array of zeros, the context showing in each case the dimensions of the zero array. Thus, in the first symbolic row the symbol (0) has as many rows as each of the horizontally collinear arrays $(\lambda), (\mu), (\lambda')$, that is, it has r rows, and it has as many columns as the vertically collinear array (μ').

4. Expanding the determinant (1) by Laplace's theorem in terms of the first r rows, the first term of the expansion is the product of the two r^{th} order determinants

$$|(\lambda)\ (\mu)|, \quad |(\lambda')\ (\mu')|.$$

It will be convenient to denote this product by S or Σ. Any other term in the Laplace expansion may be derived from S or Σ by interchanging one or more columns of the array (λ') with one or more columns of the array (μ) and changing the sign of the result when the number of columns interchanged is odd. If s, s' are the numbers of columns in the arrays (λ), (λ'), and, consequently, $r-s$, $r-s'$ are the numbers of columns in the arrays (μ), and (μ'), then the number of terms which can be derived in this way from S or Σ is

$$(r-s)_1 s'_1 + (r-s)_2 s'_2 + \dots\dots + (r-s)_t s'_t,$$

where t is the smaller of the two, $r-s$, s', and $(r-s)_n$, s'_n denote the numbers of combinations of $r-s$, s' things n together. If the sum of all terms so derived from S or Σ be denoted by ${}_\mu\Sigma_{\lambda'}$, the expansion of the determinant (1) is

$$\Sigma + {}_\mu\Sigma_{\lambda'}.$$

5. To find another equivalent for the determinant (1), subtract rows $1, 2, \dots\dots r$ from rows $r+1, r+2, \dots\dots 2r$ respectively. Thus (1) becomes

$$\begin{vmatrix} (\ \ \lambda) & (\mu) & (\lambda') & (0) \\ (-\lambda) & (0) & (0) & (\mu') \end{vmatrix}, \qquad (2)$$

where $(-\lambda)$ denotes the array formed from the array (λ) by changing the sign of every element.

If, now, the set λ' be more numerous than the set λ, the determinant (2) obviously vanishes, for the array represented by the symbols (0), (0) in the second symbolical row consists of r rows and of more than r columns.

If the two sets λ, λ' are equally numerous, then, by s interchanges of columns, and by s changes of sign, the determinant (2) takes the form

$$\begin{vmatrix} (\lambda') & (\mu) & (-\lambda) & (0) \\ (0) & (0) & (\ \ \lambda) & (\mu') \end{vmatrix},$$

and, therefore, has the value

$$|(\lambda')\ (\mu)| \cdot |(\lambda)\ (\mu')|,$$

which may be derived from S or Σ by interchanging λ, λ'.

Finally, if the set λ be more numerous than the set λ', it will be seen that the determinant (2) is the sum of the results of interchanging in S or Σ the set λ' with every equally numerous set contained in λ.

6. Denoting the sum of these results by ${}_{\lambda}S_{\lambda'}$, and equating the two expansions of the determinant (1), we have

$$\Sigma + {}_{\mu}\Sigma_{\lambda'} = {}_{\lambda}S_{\lambda'},$$

where ${}_{\lambda}S_{\lambda'}$ has the value zero, when the set λ' is more numerous than the set λ.

The result which has been obtained may be stated as follows:

If, in the product S or Σ of any two determinants

$$(\alpha\beta\gamma \ldots\ldots x), \quad (\alpha'\beta'\gamma' \ldots\ldots x')$$

of the same order, we divide the columns α, β, γ, ... x into two complementary sets λ, μ, and if we also divide the columns α', β', γ' .... x' into two complementary sets λ', μ', then denoting by ${}_{\lambda}S_{\lambda'}$ the sum of the results of interchanging the set λ', with every equally numerous set from λ and denoting by ${}_{\mu}\Sigma_{\lambda'}$ the sum of the results of interchanging one or more of the set λ' with one or more of the set μ, subject to the condition, in the latter case, that when the number of columns interchanged is odd, the sign of the determinant product is changed, we have

$$S = \Sigma = {}_{\lambda}S_{\lambda'} - {}_{\mu}\Sigma_{\lambda'}.$$

7. From this theorem it immediately follows that

If K or M be any minor

$$\begin{pmatrix} \alpha & \beta & \gamma & \ldots\ldots x \\ \alpha' & \beta' & \gamma' & \ldots\ldots x' \end{pmatrix}$$

of an axi-symmetric determinant of odd or even order, and if the row suffixes α, β, γ, ... x be divided into two complementary sets λ, μ, and also the column *suffixes* α', β', γ', ... $\varkappa'$ *into two complementary sets* λ', μ', *then*

$$K = M = {}_\lambda K_{\lambda'} - {}_\mu M_{\lambda'},$$

where ${}_\lambda K_{\lambda'}$ *denotes the sum of the results of interchanging in* K *or* M *the set* λ' *with every equally numerous set from* λ, *and* ${}_\mu M_{\lambda'}$ *denotes the sum of the results of interchanging in* K *or* M *one or more of the set* λ' *with one or more of the set* μ *and changing the sign of the minor when the number of suffixes interchanged is odd.*

8. The expression ${}_\lambda K_{\lambda'}$ may be called a Kronecker expression and the expression $K - {}_\lambda K_{\lambda'}$ may be called a Kronecker aggregate. Also, the expression ${}_\mu M_{\lambda'}$ may be called a Metzler expression and the expression $M + {}_\mu M_{\lambda'}$ may be called a Metzler aggregate. It is also convenient to speak of ${}_\mu M_{\lambda'}$ as the Metzler expression complementary to the Kronecker expression ${}_\lambda K_{\lambda'}$. We then have the theorem that

Any minor of an axi-symmetric determinant is equal to a Kronecker expression diminished by the complementary Metzler expression.

9. In order to compare this theorem with results previously known, suppose that λ, μ, λ', μ' consist of s, t, s', t' suffixes respectively, so that

$$s + t = s' + t' = r,$$

where r is the order of the minor. First, let $t = 0$, then the Metzler expression vanishes, and we have the theorem that any minor of an axi-symmetric determinant is equal to the sum of the minors derived from it by interchanging a fixed set of column suffixes with every equally numerous set of row suffixes. This result may also be expressed by saying that a Kronecker aggregate vanishes when *all* the row suffixes are subject to interchange, and was published by the present writer in the Messenger of Mathematics for January, 1902. When $s' = 1$, that is, when the fixed set of row suffixes consists of a single suffix, the result reduces to the linear relation originally given by Kronecker.

10. Next, let $s' > s$, then the Kronecker aggregate vanishes, and we have the theorem that any minor of an axi-symmetric determinant is equal to the sum of the minors derived from it by interchanging one or more of a fixed set of column suffixes with one or more of a fixed set of row suffixes, provided, first that the sign of every derived minor is changed when the number of suffixes interchanged is even, and provided, second, that the total number of suffixes subject to interchange exceeds the order of the minors involved. This result may also be expressed by saying that a Metzler aggregate vanishes whenever the total number of suffixes subject to interchange exceeds the order of the minors involved. Of this theorem, two special cases have hitherto been given. First, when all the row suffixes are subject to interchange, a Metzler aggregate, as here defined, is identical with a Metzler aggregate as defined by Muir, l. c., and vanishes, as shown by Metzler. Second, when the total number of suffixes subject to interchange exceeds the order of the minors involved by one unit, then a Metzler aggregate, as here defined, is identical with a Metzler sub-aggregate as defined by Muir, and vanishes as stated by Muir, l. c., p. 416. Finally, when $s = r - 1$, $s' = r$, the Metzler aggregate reduces to the vanishing aggregate originally given by Kronecker.

11. When $s = s'$ and, consequently, $t = t'$, the general theorem takes an interesting form. The Kronecker expression then reduces to a single term, viz. the determinant obtained by interchanging the sets λ, λ'. This single term is, therefore, equal to a Metzler aggregate, in which the total number of interchangeable suffixes is equal to the order of the minors involved. This result agrees with those indicated by Muir, l. c., p. 415, and may be expressed in the form

$$\begin{pmatrix} \lambda' & \mu \\ \lambda & \mu' \end{pmatrix} = \Sigma \begin{pmatrix} \lambda & \bar{\mu} \\ \underline{\lambda'} & \mu' \end{pmatrix},$$

where the first member denotes the minor formed with the rows λ', μ and the columns λ, μ' of an axi-symmetric determinant, the sets λ, λ' and, consequently, also the sets μ, μ' being equally numerous. The first term in the second member is the minor formed from the first member by interchanging the sets λ, λ', and the rest of the terms are derived from the first term by interchanging, as indicated by the horizontal bars used by Muir, one or more of the set λ' with one or more of the sets μ, the sign being changed whenever the number of interchanges is odd. But the theorem thus obtained is readily seen to be at once derivable from the theorem already mentioned as having been given in the Messenger of Mathematics. For, if we start with the minor $\begin{pmatrix}\lambda' & \mu \\ \lambda & \mu'\end{pmatrix}$, then the Kronecker expression derived therefrom by interchanging the fixed set λ' with every equally numerous set which can be formed from the combined set λ, μ' is obviously identical with the expression $\Sigma\begin{pmatrix}\lambda & \bar{\mu} \\ \underline{\lambda'} & \mu'\end{pmatrix}$ already defined.

MELBOURNE, *August*, 1903.